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Some properties of sharp Hessenberg pairs

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ABSTRACT

Let $\mathbb{F}$ denote a field and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. We consider an ordered pair of $\mathbb{F}$-linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following conditions: (i) each of A, A^* is diagonalizable on V; (ii) there exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that $A^*V_i \subseteq V_0 + V_1 + \cdots + V_{i+1}$ for $0 \leqslant i \leqslant d$, where $V_{-1} := 0$ and $V_{d+1} := 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that $AV_i^* \subseteq V_0^* + V_1^* + \cdots + V_{i+1}^*$ for $0 \leqslant i \leqslant \delta$, where $V_{-1}^* := 0$ and $V_{\delta+1}^* := 0$. We call such a pair a Hessenberg pair on V. It is known that if the Hessenberg pair A, A^* on V is irreducible then $d = \delta$ and for $0 \leqslant i \leqslant d$ the dimensions of V_i and V_{d-i}^* coincide. We say a Hessenberg pair A, A^* on V is sharp whenever it is irreducible and $\dim V_0^* = \dim V_d = 1$.

In this paper, we give the definitions of a Hessenberg system and a sharp Hessenberg system. We discuss the connection between a Hessenberg pair and a Hessenberg system. We also define a finite sequence of scalars called the parameter array for a sharp Hessenberg system, which consists of the eigenvalue sequence, the dual eigenvalue sequence and the split sequence. We calculate the split sequence of a sharp Hessenberg system. We show that a sharp Hessenberg pair is a tridiagonal pair if and only if there exists a nonzero nondegenerate bilinear form $\langle\ ,\ \rangle$ on V that satisfies $\langle Au, v\rangle = \langle u, Av\rangle$ and $\langle A^*u, v\rangle = \langle u, A^*v\rangle$ for all $u, v \in V$.

1. Introduction

We begin with the following situation in linear algebra. Throughout the paper $\mathbb{F}$ denotes a field and V denotes a vector space over $\mathbb{F}$ with finite positive dimension.

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By a *decomposition* of V, we mean a sequence $\{V_i\}_{i=0}^d$ consisting of nonzero subspaces of V such that $V = \sum_{i=0}^d V_i$ (direct sum). For notational convenience we set $V_{-1} := 0$, $V_{d+1} := 0$.

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V. By the *shape* of this decomposition we mean the sequence $\{\rho_i\}_{i=0}^d$ where ρ_i is the dimension of V_i for $0 \leqslant i \leqslant d$.

By a *linear transformation* on V, we mean an $\mathbb{F}$-linear map from V to V. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra consisting of all linear transformations from V to V.

Let A denote a linear transformation on V. By an *eigenspace* of A, we mean a nonzero subspace W of V of the form

$$W = \{v \in V | Av = \theta v\},$$

where $\theta \in \mathbb{F}$. In this case, we call θ the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1 [*1, Definition 1.1*]**.** By a *Hessenberg pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i)–(iii) below:

(i) Each of A, A^* is diagonalizable on V.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_0 + V_1 + \cdots + V_{i+1} \quad (0 \leqslant i \leqslant d), \tag{1}$$

where $V_{-1} := 0$ and $V_{d+1} := 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_0^* + V_1^* + \cdots + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta), \tag{2}$$

where $V_{-1}^* := 0$ and $V_{\delta+1}^* := 0$.

Let A, A^* denote an ordered pair of diagonalizable linear transformations on V. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^\delta$) denote any ordering of the eigenspaces of A (resp. A^*). We say that the pair A, A^* is Hessenberg with respect to $(\{V_i\}_{i=0}^d; \ \{V_i^*\}_{i=0}^\delta)$ whenever these orderings satisfy (1) and (2). Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^\delta$) denote the ordering of the eigenvalues of A (resp. A^*) that corresponds to $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^\delta$). We say that the pair A, A^* is Hessenberg with respect to $(\{\theta_i\}_{i=0}^d; \ \{\theta_i^*\}_{i=0}^\delta)$ whenever the pair A, A^* is Hessenberg with respect to $(\{V_i\}_{i=0}^d; \ \{V_i^*\}_{i=0}^\delta)$.

Definition 1.2 [*1, Definition 1.4*]**.** Let A, A^* denote an ordered pair of linear transformations on V. We say the pair A, A^* is *irreducible* whenever there is no subspace W of V such that $AW \subseteq W, \ A^*W \subseteq W, \ W \neq 0, \ W \neq V$.

Definition 1.3. Let A, A^* be a Hessenberg pair on V. It is called an *irreducible Hessenberg pair*, whenever it is irreducible in the sense of Definition 1.2.

For an irreducible Hessenberg pair A, A^*, the scalars d and δ from Definition 1.1 are equal by [1, Proposition 2.4] and for $0 \leqslant i \leqslant d$ the dimensions of V_{d-i} and V_i^* are the same by [1, Corollary 3.6].

Definition 1.4. Let A, A^* be a Hessenberg pair on V. It is called a *sharp Hessenberg pair*, whenever it is irreducible and both the dimensions of V_0^* and V_d are 1.

In this paper, we give the definitions of a Hessenberg system and a sharp Hessenberg system. We discuss the connection between a Hessenberg pair and a Hessenberg system. We also define a finite sequence of scalars called the parameter array for a sharp Hessenberg system, which consists of the eigenvalue sequence, the dual eigenvalue sequence and the split sequence. We calculate the split sequence of a sharp Hessenberg system. Let V' be a finite dimensional vector space with $\dim(V) = \dim(V')$, and let σ denote an anti-isomorphism from $\mathrm{End}(V)$ to $\mathrm{End}(V')$. We show that if $\Phi = \{A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta}\}$ is a Hessenberg system on V, then $\Phi^\sigma = \{A^\sigma; \{E_{d-i}^\sigma\}_{i=0}^{d}; A^{*\sigma}; \{E_{\delta-i}^{*\sigma}\}_{i=0}^{\delta}\}$ is a Hessenberg system on V'. We show that a sharp Hessenberg pair is a tridiagonal pair (see Definition 5.1) if and only if there exists a nonzero nondegenerate bilinear form $\langle\ ,\ \rangle$ on V that satisfies $\langle Au, v\rangle = \langle u, Av\rangle$ and $\langle A^*u, v\rangle = \langle u, A^*v\rangle$ for all $u, v \in V$.

2. Hessenberg systems

We first give some concepts from linear algebra. Let A denote a diagonalizable element of $\mathrm{End}(V)$. Let $\{V_i\}_{i=0}^{d}$ denote an ordering of the eigenspaces of A and let $\{\theta_i\}_{i=0}^{d}$ denote the corresponding ordering of the eigenvalues of A. For $0 \leqslant i \leqslant d$, let $E_i : V \to V$ denote the linear transformation such that $(E_i - I)V_i = 0$ and $E_iV_j = 0$ for $j \neq i$ $(0 \leqslant j \leqslant d)$. Here I denotes the identity of $\mathrm{End}(V)$. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $E_0 + E_1 + \cdots + E_d = I$; (ii) $E_iE_j = \delta_{ij}E_i$ $(0 \leqslant i, j \leqslant d)$; (iii) $E_iV = V_i$ $(0 \leqslant i \leqslant d)$; (iv) $A = \sum_{i=0}^{d} \theta_i E_i$. Moreover

$$E_i = \prod_{0\leqslant j\leqslant d, j\neq i} \frac{A - \theta_j I}{\theta_i - \theta_j}. \tag{3}$$

For $0 \leqslant i \leqslant d$, E_iV is the eigenspace of A associated with the eigenvalue θ_i. We note that $\{E_i\}_{i=0}^{d}$ is a basis for the $\mathbb{F}$-subalgebra of $\mathrm{End}(V)$ generated by A. We now define a Hessenberg system.

Definition 2.1. By a *Hessenberg system* on V, we mean a sequence

$$\Phi = \{A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta}\}$$

that satisfies (i)–(v) below.

(i) A, A^* are both diagonalizable linear transformations on V.
(ii) $\{E_i\}_{i=0}^{d}$ is an ordering of the primitive idempotents of A.
(iii) $\{E_i^*\}_{i=0}^{\delta}$ is an ordering of the primitive idempotents of A^*.
(iv) $E_iA^*E_j = 0$ if $i > j + 1$ $(0 \leqslant i, j \leqslant d)$.
(v) $E_i^*AE_j^* = 0$ if $i > j + 1$ $(0 \leqslant i, j \leqslant \delta)$.

For notational convenience we set $E_{-1} := 0$, $E_{d+1} := 0$, $E_{-1}^* := 0$, $E_{\delta+1}^* := 0$.

In the following two lemmas, we give the connection between Hessenberg pairs and Hessenberg systems.

Lemma 2.2. *Let A, A^* denote an ordered pair of diagonalizable linear transformations on V. Assume the ordered pair A, A^* is Hessenberg with respect to $(\{V_i\}_{i=0}^{d};\ \{V_i^*\}_{i=0}^{\delta})$. Let E_i be the primitive idempotent of A corresponding to V_i for $0 \leqslant i \leqslant d$ and let E_i^* be the primitive idempotent of A^* corresponding to V_i^* for $0 \leqslant i \leqslant \delta$. Then*

$$\Phi = \{A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta}\}$$

is a Hessenberg system on V.

Proof. We verify the conditions (i)–(v) of Definition 2.1. Condition (i) is just Definition 1.1(i). Conditions (ii)–(iii) are obvious by assumption. To obtain condition (iv), pick any integers i, j $(0 \leqslant i, j \leqslant d)$, and assume $i > j + 1$. For all $v \in V$,

$$\begin{aligned} E_iA^*E_jv &\in E_iA^*V_j \\ &\subseteq E_i(V_0 + V_1 + \cdots + V_{j+1}) \\ &= 0, \end{aligned}$$

so $E_iA^*E_j = 0$. We have now verified condition (iv), and condition (v) is similarly obtained. □

Lemma 2.3. *Let*

$$\Phi = \{A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta\}$$

denote a Hessenberg system on V. Then

$$A^*E_iV \subseteq E_0V + E_1V + \cdots + E_{i+1}V \quad (0 \leqslant i \leqslant d), \tag{4}$$

$$AE_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i+1}^*V \quad (0 \leqslant i \leqslant \delta). \tag{5}$$

Moreover, the ordered pair A, A^* is Hessenberg with respect to ($\{V_i\}_{i=0}^d$; $\{V_i^*\}_{i=0}^\delta$), where $V_i = E_iV$ for $0 \leqslant i \leqslant d$ and $V_i^* = E_i^*V$ for $0 \leqslant i \leqslant \delta$. We refer to A, A^* as the Hessenberg pair associated with Φ.

Proof. To obtain (4), observe by the equation $E_0 + E_1 + \cdots + E_d = I$ and Definition 2.1(iv) that

$$\begin{aligned} A^*E_iV &= (E_0 + E_1 + \cdots + E_d)A^*E_iV \\ &= (E_0 + E_1 + \cdots + E_{i+1})A^*E_iV \\ &\subseteq E_0V + E_1V + \cdots + E_{i+1}V. \end{aligned}$$

Line (5) is similarly obtained, and the last assertion follows. □

Definition 2.4. Let $\Phi = \{A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta\}$ denote a Hessenberg system on V. Let θ_i denote the eigenvalue of A associated with the eigenspace E_iV for $0 \leqslant i \leqslant d$ and let θ_i^* denote the eigenvalue of A^* associated with the eigenspace E_i^*V for $0 \leqslant i \leqslant \delta$. We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^\delta$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ. We observe $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^\delta$) are mutually distinct and contained in $\mathbb{F}$.

Definition 2.5. Let $\Phi = \{A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta\}$ denote a Hessenberg system on V. We say Φ is an *irreducible Hessenberg system*, whenever the ordered pair A, A^* is an irreducible Hessenberg pair. We say Φ is a *sharp Hessenberg system*, whenever the ordered pair A, A^* is a sharp Hessenberg pair.

Lemma 2.6. *With reference to Definition 2.4, the following* (i)–(ii) *hold.*

(i) $E_iA^{*k}E_j = 0$ *if* $i > j + k \quad (0 \leqslant i, j, k \leqslant d)$.
(ii) $E_i^*A^kE_j^* = 0$ *if* $i > j + k \quad (0 \leqslant i, j, k \leqslant \delta)$.

Proof. Routinely obtained using Definition 2.1 (iv) and (v). □

When discussing sharp Hessenberg systems we will use the following notational convention. Let λ denote an indeterminate and let $\mathbb{F}[\lambda]$ denote the $\mathbb{F}$-algebra consisting of all polynomials in λ that have coefficients in $\mathbb{F}$. With reference to Definition 2.4, for $0 \leqslant i \leqslant d$ and $0 \leqslant j \leqslant \delta$ we define the following polynomials in $\mathbb{F}[\lambda]$:

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1)\cdots(\lambda - \theta_{i-1}),$$
$$\eta_i = (\lambda - \theta_d)(\lambda - \theta_{d-1})\cdots(\lambda - \theta_{d-i+1}),$$
$$\tau_j^* = (\lambda - \theta_0^*)(\lambda - \theta_1^*)\cdots(\lambda - \theta_{j-1}^*),$$
$$\eta_j^* = (\lambda - \theta_\delta^*)(\lambda - \theta_{\delta-1}^*)\cdots(\lambda - \theta_{\delta-j+1}^*).$$

Note that each of τ_i, η_i is monic with degree i and each of τ_j^*, η_j^* is monic with degree j. By (3), for $0 \leqslant i \leqslant d$ and $0 \leqslant j \leqslant \delta$ we have

$$E_i = \frac{\tau_i(A)\eta_{d-i}(A)}{\tau_i(\theta_i)\eta_{d-i}(\theta_i)}, \quad E_j^* = \frac{\tau_j^*(A^*)\eta_{\delta-j}^*(A^*)}{\tau_j^*(\theta_j^*)\eta_{\delta-j}^*(\theta_j^*)}.$$

In particular

$$E_0 = \frac{\eta_d(A)}{\eta_d(\theta_0)}, \quad E_d = \frac{\tau_d(A)}{\tau_d(\theta_d)}, \tag{6}$$

$$E_0^* = \frac{\eta_\delta^*(A^*)}{\eta_\delta^*(\theta_0^*)}, \quad E_\delta^* = \frac{\tau_\delta^*(A^*)}{\tau_\delta^*(\theta_\delta^*)}. \tag{7}$$

We mention a result for future use.

Lemma 2.7 [3, Proposition 5.5]**.** *With reference to Definition 2.4,*

$$\eta_d = \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\tau_i, \quad \eta_\delta^* = \sum_{i=0}^{\delta} \eta_{\delta-i}^*(\theta_0^*)\tau_i^*, \tag{8}$$

$$\tau_d = \sum_{i=0}^{d} \tau_{d-i}(\theta_d)\eta_i, \quad \tau_\delta^* = \sum_{i=0}^{\delta} \tau_{\delta-i}^*(\theta_\delta^*)\eta_i^*. \tag{9}$$

3. The split decomposition and the parameter array

We first recall the split decomposition. We start with a comment. With reference to Definition 2.4, assume $\Phi = \{A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\}$ is irreducible. For $0 \leqslant i \leqslant d$ define

$$U_i = (E_0V + E_1V + \cdots + E_{d-i}V) \cap (E_0^*V + E_1^*V + \cdots + E_i^*V). \tag{10}$$

By [1, Lemma 2.5] the sequence $\{U_i\}_{i=0}^d$ is a decomposition of V. By [1, Lemma 3.2], both

$$U_0 + U_1 + \cdots + U_i = E_0^*V + E_1^*V + \cdots + E_i^*V, \tag{11}$$
$$U_i + U_{i+1} + \cdots + U_d = E_0V + E_1V + \cdots + E_{d-i}V \tag{12}$$

for $0 \leqslant i \leqslant d$. By [1, Corollary 3.6] the dimensions of U_i, $E_{d-i}V$ and E_i^*V are the same. By [1, Lemma 2.3] both

$$(A - \theta_{d-i}I)U_i \subseteq U_{i+1}, \tag{13}$$

$$(A^* - \theta_i^*I)U_i \subseteq U_{i-1} \tag{14}$$

for $0 \leqslant i \leqslant d$, where $U_{-1} := 0$ and $U_{d+1} := 0$. The sequence $\{U_i\}_{i=0}^d$ is called the Φ-*split decomposition* of V.

Let $\Phi = \{A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\}$ denote a sharp Hessenberg system on V and let $\{U_i\}_{i=0}^d$ be the Φ-split decomposition of V. Note that $U_0 (= E_0^* V)$ has dimension 1 and for $0 \leqslant i \leqslant d$ the space U_0 is invariant under

$$(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)(A - \theta_{d-i+1} I) \cdots (A - \theta_{d-1} I)(A - \theta_d I). \tag{15}$$

Let ξ_i denote the corresponding eigenvalue. Note that $\xi_0 = 1$. We call the sequence $\{\xi_i\}_{i=0}^d$ the *split sequence* of Φ.

Definition 3.1. With reference to Definition 2.4, assume Φ is a sharp Hessenberg system. By the *parameter array* of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\xi_i\}_{i=0}^d)$, where $\{\theta_i\}_{i=0}^d$ is the eigenvalue sequence of Φ, $\{\theta_i^*\}_{i=0}^d$ is the dual eigenvalue sequence of Φ and $\{\xi_i\}_{i=0}^d$ is the split sequence of Φ.

4. Calculation of the split sequence

In this section, we calculate the split sequence of a sharp Hessenberg system. To prepare for this we have a few lemmas.

Lemma 4.1. *With reference to Definition 2.4, assume that $d = \delta$. Then for $0 \leqslant i, j \leqslant d$ we have*

$$E_d \tau_i^*(A^*) \eta_j(A) E_0^* = 0 \tag{16}$$

provided $i \neq j$.

Proof. We first show (16) for $i < j$. In the left-hand side of (16) we insert a factor I between $\tau_i^*(A^*)$ and $\eta_j(A)$. We expand using $I = \sum_{r=0}^d E_r$ and simplify the result using $E_r \eta_j(A) = \eta_j(\theta_r) E_r$ for $0 \leqslant r \leqslant d$. By these comments the left-hand side of (16) is equal to

$$\sum_{r=0}^{d} \eta_j(\theta_r) E_d \tau_i^*(A^*) E_r E_0^*. \tag{17}$$

For $0 \leqslant r \leqslant d$, we examine term r in (17). By the definition of η_j, we have $\eta_j(\theta_r) = 0$ for $d - j + 1 \leqslant r \leqslant d$. By Lemma 2.6(i), we find $E_d A^{*s} E_r = 0$ for $0 \leqslant r < d - s$. By this and since $\tau_i^*(A^*)$ has degree i, we find $E_d \tau_i^*(A^*) E_r = 0$ for $0 \leqslant r < d - i$. By these comments term r in (17) vanishes for $d - j + 1 \leqslant r \leqslant d$ and $0 \leqslant r < d - i$. Recall $i < j$ so term r in (17) vanishes for $0 \leqslant r \leqslant d$. In other words (17) is 0. We have shown (16) for $i < j$. Next we show (16) for $i > j$. In the left-hand side of (16) we again insert a factor I between $\tau_i^*(A^*)$ and $\eta_j(A)$. This time we expand using $I = \sum_{r=0}^d E_r^*$ and simplify the result using $\tau_i^*(A^*) E_r^* = \tau_i^*(\theta_r^*) E_r^*$ for $0 \leqslant r \leqslant d$. By these comments the left-hand side of (16) is equal to

$$\sum_{r=0}^{d} \tau_i^*(\theta_r^*) E_d E_r^* \eta_j(A) E_0^*. \tag{18}$$

For $0 \leqslant r \leqslant d$, we examine term r in (18). By the definition of τ_i^*, we have $\tau_i^*(\theta_r^*) = 0$ for $0 \leqslant r \leqslant i-1$. By Lemma 2.6(ii), we find $E_r^* A^s E_0^* = 0$ for $s < r \leqslant d$. By this and since $\eta_j(A)$ has degree j, we find $E_r^* \eta_j(A) E_0^* = 0$ for $j < r \leqslant d$. By these comments term r in (18) vanishes for $0 \leqslant r \leqslant i - 1$ and $j < r \leqslant d$. Recall $i > j$ so term r in (18) vanishes for $0 \leqslant r \leqslant d$. In other words (18) is 0. We have shown (16) for $i > j$. $\square$

Lemma 4.2. *With reference to Definition 2.4, assume that $d = \delta$. Then for $0 \leqslant i \leqslant d$ both*

$$E_d\tau_i^*(A^*)\eta_i(A)E_0^* = (\theta_d - \theta_{d-1})(\theta_d - \theta_{d-2})\cdots(\theta_d - \theta_{d-i})E_d\tau_i^*(A^*)E_dE_0^*, \tag{19}$$

$$E_d\tau_i^*(A^*)\eta_i(A)E_0^* = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)\cdots(\theta_0^* - \theta_i^*)E_dE_0^*\eta_i(A)E_0^*. \tag{20}$$

Proof. In the expression on the right in (19), eliminate the middle E_d using the equation on the right in (6); evaluate the result using (9) and (16) to get the expression on the left in (19). This gives (19). Line (20) is similarly obtained. $\square$

We now restrict our attention to sharp Hessenberg systems.

Lemma 4.3. *With reference to Definition 2.4, assume Φ is a sharp Hessenberg system. Let $\{\xi_i\}_{i=0}^d$ denote the split sequence of Φ. Then for $0 \leqslant i \leqslant d$ both*

$$(A^* - \theta_1^*I)(A^* - \theta_2^*I)\cdots(A^* - \theta_i^*I)\eta_i(A)E_0^* = \xi_iE_0^*, \tag{21}$$

$$E_d\tau_i^*(A^*)\eta_i(A)E_0^* = \xi_iE_dE_0^*. \tag{22}$$

Proof. By the construction

$$(A^* - \theta_1^*I)(A^* - \theta_2^*I)\cdots(A^* - \theta_i^*I)\eta_i(A) - \xi_iI$$

vanishes on U_0 $(= E_0^*V)$, so (21) holds. Next, we show (22). Multiplying both sides of (21) on the left by E_d,

$$E_d(A^* - \theta_1^*I)(A^* - \theta_2^*I)\cdots(A^* - \theta_i^*I)\eta_i(A)E_0^* = \xi_iE_dE_0^*. \tag{23}$$

Observe that the expression

$$(A^* - \theta_1^*I)(A^* - \theta_2^*I)\cdots(A^* - \theta_i^*I) - \tau_i^*(A^*)$$

is a polynomial in A^* with degree less than i, so it is a linear combination of $\{\tau_r^*(A^*)\}_{r=0}^{i-1}$. By this and Lemma 4.1 we find that the left-hand side of (23) is equal to the left-hand side of (22). This gives (22). $\square$

Theorem 4.4. *With reference to Definition 2.4, assume Φ is a sharp Hessenberg system. Let $\{\xi_i\}_{i=0}^d$ denote the split sequence of Φ. Then for $0 \leqslant i \leqslant d$*

$$\xi_i = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)\cdots(\theta_0^* - \theta_i^*)\mathrm{tr}(\eta_i(A)E_0^*). \tag{24}$$

Moreover if $\mathrm{tr}(E_dE_0^*)$ is not zero, then for $0 \leqslant i \leqslant d$, both

$$\xi_i = (\theta_d - \theta_{d-1})(\theta_d - \theta_{d-2})\cdots(\theta_d - \theta_{d-i})\mathrm{tr}(E_d\tau_i^*(A^*)), \tag{25}$$

$$\xi_i = \frac{\mathrm{tr}(E_d\tau_i^*(A^*)\eta_i(A)E_0^*)}{\mathrm{tr}(E_dE_0^*)}. \tag{26}$$

Proof. First, we take the trace of both sides of (21) and use $\mathrm{tr}(MN) = \mathrm{tr}(NM)$, $\mathrm{tr}(E_0^*) = 1$, $E_0^*A^* = \theta_0^*E_0^*$ to find (24). Next, assume that $\mathrm{tr}(E_dE_0^*)$ is not zero. We take the trace of both sides of (22) to find (26). To show (25), observe that the left hand sides of (19) and (20) coincide. Since E_d has rank 1 we find $E_dE_0^*E_d$ is a scalar multiple of E_d. Taking the trace we find $E_dE_0^*E_d = \mathrm{tr}(E_dE_0^*)E_d$. Using this and $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ we find that the trace of the right hand side of (19) is equal to the right hand side of

(25) times $\mathrm{tr}(E_d E_0^*)$. Similarly, the trace of the right hand side of (20) is equal to the right hand side of (24) times $\mathrm{tr}(E_d E_0^*)$, i.e., $\xi_i \mathrm{tr}(E_d E_0^*)$. By these comments and the assumption that $\mathrm{tr}(E_d E_0^*)$ is not zero, we get (25). $\square$

5. Hessenberg pairs and tridiagonal pairs

In this section, we explain how Hessenberg pairs are related to tridiagonal pairs. We start by recalling the notion of a tridiagonal pair.

Definition 5.1 *[2, Definition 1.1]***.** By a *tridiagonal pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

(i) Each of A, A^* is diagonalizable on V.
(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leqslant i \leqslant d), \tag{27}$$

where $V_{-1} := 0$ and $V_{d+1} := 0$.
(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta), \tag{28}$$

where $V_{-1}^* := 0$ and $V_{\delta+1}^* := 0$.
(iv) The pair A, A^* is irreducible in the sense of Definition 1.2.

Observe that a tridiagonal pair A, A^* on V is an irreducible Hessenberg pair, but the converse is not true. In order to give a criterion for an irreducible Hessenberg pair to be a tridiagonal pair, we recall the notion of bilinear forms. For more information on bilinear forms, see [5].

Throughout this section let V' denote a vector space over $\mathbb{F}$ such that $\dim V' = \dim V$.

A map $\langle\, ,\, \rangle : V \times V' \to \mathbb{F}$ is called a *bilinear form* whenever the following conditions hold for $u, v \in V$, for $u', v' \in V'$, and for $\alpha \in \mathbb{F}$: (i) $\langle u + v,\ u' \rangle = \langle u,\ u' \rangle + \langle v,\ u' \rangle$; (ii) $\langle \alpha u,\ u' \rangle = \alpha \langle u,\ u' \rangle$; (iii) $\langle u,\ u' + v' \rangle = \langle u,\ u' \rangle + \langle u,\ v' \rangle$; (iv) $\langle u,\ \alpha u' \rangle = \alpha \langle u,\ u' \rangle$. Let $\langle\, ,\, \rangle : V \times V' \to \mathbb{F}$ denote a bilinear form. Then the following are equivalent: (i) there exists a nonzero $v \in V$ such that $\langle v,\ v' \rangle = 0$ for all $v' \in V'$; (ii) there exists a nonzero $v' \in V'$ such that $\langle v,\ v' \rangle = 0$ for all $v \in V$. The form is said to be *degenerate* whenever (i), (ii) hold and *nondegenerate* otherwise. By a bilinear form on V we mean a bilinear form $\langle\, ,\, \rangle : V \times V \to \mathbb{F}$.

By an $\mathbb{F}$-*algebra anti-isomorphism* from $\mathrm{End}(V)$ to $\mathrm{End}(V')$ we mean an isomorphism of $\mathbb{F}$-vector spaces $\sigma : \mathrm{End}(V) \to \mathrm{End}(V')$ such that $(XY)^\sigma = Y^\sigma X^\sigma$ for all $X, Y \in \mathrm{End}(V)$. By an *anti-automorphism* of $\mathrm{End}(V)$ we mean an $\mathbb{F}$-algebra anti-isomorphism from $\mathrm{End}(V)$ to $\mathrm{End}(V)$.

Let $\langle\, ,\, \rangle : V \times V' \to \mathbb{F}$ denote a nondegenerate bilinear form. Then there exists a unique anti-isomorphism $\sigma : \mathrm{End}(V) \to \mathrm{End}(V')$ such that $\langle Xv,\ v' \rangle = \langle v,\ X^\sigma v' \rangle$ for all $v \in V, v' \in V'$ and $X \in \mathrm{End}(V)$. Conversely, given an anti-isomorphism $\sigma : \mathrm{End}(V) \to \mathrm{End}(V')$, there exists a bilinear form $\langle\, ,\, \rangle : V \times V' \to \mathbb{F}$ such that $\langle Xv,\ v' \rangle = \langle v,\ X^\sigma v' \rangle$ for all $v \in V, v' \in V'$ and $X \in \mathrm{End}(V)$. This form is nondegenerate and uniquely determined by σ up to multiplication by a nonzero scalar in $\mathbb{F}$. We say the form $\langle\, ,\, \rangle$ is *associated with* σ.

Lemma 5.2. *With reference to Definition 2.4, let V' be as above stated and let σ denote an anti-isomorphism from $\mathrm{End}(V)$ to $\mathrm{End}(V')$. Then $\Phi^\sigma = \{A^\sigma; \{E_{d-i}^\sigma\}_{i=0}^d; A^{*\sigma}; \{E_{\delta-i}^{*\sigma}\}_{i=0}^\delta\}$ is a Hessenberg system on V'.*

Proof. Let $\Phi = \{A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta\}$ denote a Hessenberg system on V. We show that $\Phi^\sigma = \{A^\sigma; \{E_{d-i}^\sigma\}_{i=0}^d; A^{*\sigma}; \{E_{\delta-i}^{*\sigma}\}_{i=0}^\delta\}$ is a Hessenberg system on V'. To do this, we verify $\Phi^\sigma = \{A^\sigma; \{E_{d-i}^\sigma\}_{i=0}^d;$

$A^{*\sigma}$; $\{E^{*\sigma}_{\delta-i}\}_{i=0}^{\delta}\}$ satisfies the conditions (i)–(v) of Definition 2.1. Concerning Definition 2.1(i), we claim that A^{σ} is diagonalizable with eigenvalues $\{\theta_i\}_{i=0}^{d}$. For $f \in \mathbb{F}[\lambda]$ we have $f(A) = 0$ if and only if $f(A^{\sigma}) = 0$. Therefore, A^{σ} has the same minimal polynomial $\Pi_{i=0}^{d}(\lambda - \theta_i)$ as A, and $\{\theta_i\}_{i=0}^{d}$ is the eigenvalue sequence of A^{σ}. By this and since $\{\theta_i\}_{i=0}^{d}$ are mutually distinct, A^{σ} is diagonalizable with eigenvalues $\{\theta_i\}_{i=0}^{d}$. Similarly $A^{*\sigma}$ is diagonalizable with eigenvalues $\{\theta^*_i\}_{i=0}^{\delta}$. Concerning Definition 2.1(ii), for $0 \leqslant i \leqslant d$, applying σ to (3) we find E_i^{σ} is the primitive idempotent of A^{σ} associated with θ_i. By the construction, $\{E^{\sigma}_{d-i}\}_{i=0}^{d}$ is an ordering of the primitive idempotents of A^{σ}. Similarly, $\{E^{*\sigma}_{\delta-i}\}_{i=0}^{\delta}$ is an ordering of the primitive idempotents of $A^{*\sigma}$ and Definition 2.1(iii) holds. Concerning Definition 2.1(iv), applying σ to the equation in Definition 2.1(iv), we find that $E_j^{\sigma} A^{*\sigma} E_i^{\sigma} = 0$ if $i > j+1$ $(0 \leqslant i, j \leqslant d)$. Set $E'_i = E^{\sigma}_{d-i}$. We have $E'_{d-j}A^{*\sigma}E'_{d-i} = 0$ if $i > j+1$, i.e., if $d-j > d-i+1$ $(0 \leqslant i, j \leqslant d)$. Set $i' = d-j$ and $j' = d-i$, then we have $E'_{i'}A^{*\sigma}E'_{j'} = 0$ if $i' > j'+1$ $(0 \leqslant i', j' \leqslant d)$. So for the ordering $\{E^{\sigma}_{d-i}\}_{i=0}^{d}$ of the primitive idempotents of A^{σ}, we have $E^{\sigma}_{d-i}A^{*\sigma}E^{\sigma}_{d-j} = 0$ if $i > j+1$ $(0 \leqslant i, j \leqslant d)$. The proof of Definition 2.1 (v) is similar to that of Definition 2.1(iv). Thus $\Phi^{\sigma} = \{A^{\sigma}; \{E^{\sigma}_{d-i}\}_{i=0}^{d}; A^{*\sigma}; \{E^{*\sigma}_{\delta-i}\}_{i=0}^{\delta}\}$ is a Hessenberg system on V' by Definition 2.1. □

Lemma 5.3. *With reference to Definition 2.4, let V' be as above stated and let σ denote an anti-isomorphism from* $\mathrm{End}(V)$ *to* $\mathrm{End}(V')$. *If the Hessenberg system Φ is irreducible then so is Φ^{σ}.*

Proof. Let $\Phi = \{A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{\delta}\}$ denote a Hessenberg system on V. Assume the pair A, A^* is irreducible. We show that the Hessenberg system Φ^{σ} is irreducible. To do this, we verify that the pair $A^{\sigma}, A^{*\sigma}$ on V' is irreducible. Let W denote a subspace of V' such that $A^{\sigma}W \subseteq W$ and $A^{*\sigma}W \subseteq W$. We show $W = 0$ or $W = V'$. Define $W^{\perp} = \{v \in V | \langle v, w\rangle = 0 \ \text{ for all } \ w \in W\}$, where $\langle \ , \ \rangle$ is the bilinear form associated with σ. Since $\langle \ , \ \rangle$ is nondegenerate, $\dim W + \dim W^{\perp}$ is equal to the common dimension of V, V'. Observe $AW^{\perp} \subseteq W^{\perp}$, indeed for all $w \in W$ and all $v \in W^{\perp}$, $\langle Av, w\rangle = \langle v, A^{\sigma}w\rangle = 0$ since $A^{\sigma}W \subseteq W$. We similarly obtain $A^*W^{\perp} \subseteq W^{\perp}$. Now $W^{\perp} = 0$ or $W^{\perp} = V$ since the linear transformations A, A^* on V is irreducible, and thus $W = V'$ or $W = 0$. Now the pair $A^{\sigma}, A^{*\sigma}$ on V' is irreducible. □

Lemma 5.4. *With reference to Definition 2.4, assume Φ is irreducible. Let $\langle \ , \ \rangle$ denote a nonzero bilinear form on V that satisfies*

$$\langle Au, v\rangle = \langle u, Av\rangle, \quad \langle A^*u, v\rangle = \langle u, A^*v\rangle, \quad \textit{for } u, v \in V. \tag{29}$$

Then $\langle \ , \ \rangle$ is nondegenerate.

Proof. It suffices to show that the space $W = \{w \in V | \langle w, V\rangle = 0\}$ is zero. Using (29) we routinely find $AW \subseteq W$ and $A^*W \subseteq W$, so either $W = 0$ or $W = V$ by Definition 1.2. But $W \neq V$ since $\langle \ , \ \rangle$ is nonzero, so $W = 0$ as desired. □

Theorem 5.5. *Let the linear transformations A, A^* on V be an irreducible Hessenberg pair. If there exists a nonzero bilinear form $\langle \ , \ \rangle$ on V that satisfies (29) then A, A^* is a tridiagonal pair.*

Proof. Let A, A^* be two diagonalizable linear transformations on V. Assume that the pair A, A^* is irreducible and Hessenberg with respect to $(\{V_i\}_{i=0}^{d}; \{V^*_i\}_{i=0}^{d})$. Then there exists an irreducible Hessenberg system $\Phi = \{A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{d}\}$ in the sense of Lemma 2.2, where E_i (resp. E^*_i) is the primitive idempotent of A (resp. A^*) corresponding to V_i (resp. V^*_i) for $0 \leqslant i \leqslant d$. Assume that there exists a nonzero bilinear form $\langle \ , \ \rangle$ on V that satisfies (29). By Lemma 5.4 the bilinear form $\langle \ , \ \rangle$ is nondegenerate, and by our comments above Lemma 5.2, there exists an anti-automorphism σ of $\mathrm{End}(V)$, which is associated with the nonzero nondegenerate bilinear form $\langle \ , \ \rangle$. By Lemmas 5.2 and 5.3, we know that $\Phi^{\sigma} = \{A^{\sigma}; \{E^{\sigma}_{d-i}\}_{i=0}^{d}; A^{*\sigma}; \{E^{*\sigma}_{d-i}\}_{i=0}^{d}\}$ is an irreducible Hessenberg system on V. But by (29), we find $A^{\sigma} = A$ and $A^{*\sigma} = A^*$, and then by (3) we have $E_i^{\sigma} = E_i$ and $E_i^{*\sigma} = E^*_i$ for $0 \leqslant i \leqslant d$. So

$\Phi^{\sigma} = \{A; \{E_{d-i}\}_{i=0}^{d}; A^*; \{E^*_{d-i}\}_{i=0}^{d}\}$. By Lemma 2.3, the pair A, A^* on V is irreducible and Hessenberg with respect to $(\{V_{d-i}\}_{i=0}^{d}; \{V^*_{d-i}\}_{i=0}^{d})$. By [1, Proposition 4.4], A, A^* is a tridiagonal pair. $\square$

Corollary 5.6. *Let the linear transformations A, A^* on V be a sharp Hessenberg pair. Then A, A^* is a tridiagonal pair if and only if there exists a nonzero bilinear form $\langle\ ,\ \rangle$ on V that satisfies* (29).

Proof. The if part is from Theorem 5.5. The only if part is from [4, Theorem 1.4]. $\square$

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